Holomorphic foliations on complex moment-angle manifolds

based on joint works with Hiroaki Ishida, Roman Krutowski, Yuri Ustinovsky and Misha Verbitsky

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The moment-angle complex

$\mathcal{K}$ an abstract simplicial complex on the set $[m] = \{1, 2, \ldots, m\}$
$I = \{i_1, \ldots, i_k\} \in \mathcal{K}$ a simplex; always assume $\varnothing \in \mathcal{K}$.

Consider the m-dimensional unit polydisc:

$$\mathbb{D}^m = \{(z_1, \ldots, z_m) \in \mathbb{C}^m : |z_i|^2 \leqslant 1 \text{ for } i = 1, \ldots, m\}.$$

The moment-angle complex is

$$\mathcal{Z}_{\mathcal{K}} := \bigcup_{I \in \mathcal{K}} \Bigl(\prod_{i \in I} \mathbb{D} \times \prod_{i \notin I} \mathbb{S}\Bigr) \subset \mathbb{D}^m,$$

where $\mathbb{S}$ is the boundary of the unit disk $\mathbb{D}$.

$\mathcal{Z}_{\mathcal{K}}$ has a natural action of the torus T^m.
When $\mathcal{K}$ is a simplicial subdivision of sphere (e.g., the boundary of a simplicial polytope), $\mathcal{Z}_{\mathcal{K}}$ is a topological manifold, called the moment-angle manifold.

We define an open submanifold $U(\mathcal{K}) \subset \mathbb{C}^m$ in a similar way:

$$U(\mathcal{K}) := \bigcup_{I \in \mathcal{K}} \Big(\prod_{i \in I} \mathbb{C} \times \prod_{i \notin I} \mathbb{C}^\times \Big), \qquad \mathbb{C}^\times = \mathbb{C} \setminus \{0\}.$$

$U(\mathcal{K})$ is a toric variety with the corresponding fan given by

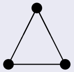
$$\Sigma_{\mathcal{K}} = \{\mathbb{R}_\geqslant \langle \mathbf{e}_i \colon i \in I \rangle \colon I \in \mathcal{K}\},$$

where $\mathbf{e}_i$ denotes the i-th standard basis vector of $\mathbb{R}^m$.

Theorem

(a) $U(\mathcal{K}) = \mathbb{C}^m \setminus \bigcup_{\{i_1,\ldots,i_k\} \notin \mathcal{K}} \{z_{i_1} = \cdots = z_{i_k} = 0\}$
(the complement to a coordinate subspace arrangement);

(b) *There is a T^m-equivariant deformation retraction $U(\mathcal{K}) \xrightarrow{\simeq} \mathcal{Z}_{\mathcal{K}}$.*

E.g., $\mathcal{K} =$  Then $U(\mathcal{K}) = \mathbb{C}^3 \setminus \{z_1 = z_2 = z_3 = 0\} \xrightarrow{\simeq} S^5 = \mathcal{Z}_{\mathcal{K}}$.

Complex-analytic structrures on moment-angle manifolds

Assume $\mathcal{Z}_\mathcal{K}$ admits a T^m-invariant complex structure.
Then the T^m-action extends to a holomorphic action of $(\mathbb{C}^\times)^m$ on $\mathcal{Z}_\mathcal{K}$.
Have a complex-analytic subgroup of global stabilisers

$$H = \{g \in (\mathbb{C}^\times)^m \colon g \cdot x = x \text{ for all } x \in \mathcal{Z}_\mathcal{K}\}.$$

$\mathfrak{h} = \mathrm{Lie}(H)$ is a complex subalgebra of $\mathrm{Lie}(\mathbb{C}^\times)^m = \mathbb{C}^m$ and satisfies

(a) the composite $\mathfrak{h} \hookrightarrow \mathbb{C}^m \xrightarrow{\mathrm{Re}} \mathbb{R}^m$ is injective;

(b) the quotient map $q \colon \mathbb{R}^m \to \mathbb{R}^m / \mathrm{Re}(\mathfrak{h})$ sends the fan $\Sigma_\mathcal{K}$ to a complete fan $q(\Sigma_\mathcal{K})$ in $\mathbb{R}^m / \mathrm{Re}(\mathfrak{h})$.

Theorem (Ishida)

Every complex moment-angle manifold $\mathcal{Z}_\mathcal{K}$ is T^m-equivariantly biholomorphic to the quotient manifold $U(\mathcal{K})/H$.

Conversely, suppose $\mathfrak{h} \subset \mathbb{C}^m$ satisfies conditions (a) and (b) above, and let H be the complex Lie subgroup of $(\mathbb{C}^\times)^m$ corresponding to $\mathfrak{h}$.

Theorem (P.-Ustinovsky)

(1) *the holomorphic action of the group $H \cong \mathbb{C}^\ell$ on $U(\mathcal{K})$ is free and proper, so the quotient $U(\mathcal{K})/H$ is a compact complex $(m-\ell)$-manifold;*

(2) *there is a T^m-equivariant diffeomorphism $U(\mathcal{K})/H \cong \mathcal{Z}_\mathcal{K}$ defining a complex structure on $\mathcal{Z}_\mathcal{K}$ in which T^m acts by holomorphic transformations.*

Thus, $\mathcal{Z}_\mathcal{K}$ admits a complex structure if and only if $\mathcal{K}$ is the underlying complex of a complete simplicial fan (that is, $\mathcal{K}$ is a star-shaped sphere triangulation), and any complex structure on such $\mathcal{Z}_\mathcal{K}$ is defined by a choice of a complex subspace $\mathfrak{h} \subset \mathbb{C}^m$ satisfying (a) and (b) above.

Example (holomorphic tori)

Let $\mathcal{K}$ be empty on 2 elements (that is, $\mathcal{K}$ has two ghost vertices). We therefore have $m = 2$, $\ell = 1$.

Let $\psi\colon \mathbb{C} \to \mathbb{C}^2$ be given by $z \mapsto (z, \alpha z)$ for some $\alpha \in \mathbb{C}$, so that

$$H = \big\{(e^z, e^{\alpha z})\big\} \subset (\mathbb{C}^\times)^2.$$

Condition (b) above is void, while (a) is equivalent to $\alpha \notin \mathbb{R}$. Then $\exp\psi\colon H \to (\mathbb{C}^\times)^2$ is an embedding, and the quotient $(\mathbb{C}^\times)^2/H$ is a complex torus $T^2_\mathbb{C}$ with parameter $\alpha \in \mathbb{C}$:

$$(\mathbb{C}^\times)^2/H \cong \mathbb{C}/(\mathbb{Z} \oplus \alpha\mathbb{Z}) = T^2_\mathbb{C}(\alpha).$$

Similarly, if $\mathcal{K}$ is empty on 2ℓ elements (so that $m = 2\ell$), we can obtain any complex torus $T^{2\ell}_\mathbb{C}$ as the quotient $(\mathbb{C}^\times)^{2\ell}/H$.

Example (Hopf manifold)

Let Σ be a complete fan in $\mathbb{R}^n$ whose cones are generated by all proper subsets of $n+1$ vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n, -\mathbf{e}_1 - \ldots - \mathbf{e}_n$.

Add one 'empty' 1-cone to make $m-n$ even: $m = n+2$, $\ell = 1$.

The underlying complex $\mathcal{K} = \partial\Delta^n$ with $n+1$ vertices and 1 ghost vertex, $\mathcal{Z}_\mathcal{K} \cong S^1 \times S^{2n+1}$, and $U(\mathcal{K}) = \mathbb{C}^\times \times (\mathbb{C}^{n+1} \setminus \{0\})$.

Take $\psi\colon \mathbb{C} \to \mathbb{C}^{n+2}$, $z \mapsto (z, \alpha z, \ldots, \alpha z)$ for some $\alpha \in \mathbb{C}$, $\alpha \notin \mathbb{R}$. Then

$$H = \bigl\{(e^z, e^{\alpha z}, \ldots, e^{\alpha z})\colon z \in \mathbb{C}\bigr\} \subset (\mathbb{C}^\times)^{n+2},$$

and $\mathcal{Z}_\mathcal{K}$ acquires a complex structure as the quotient $U(\mathcal{K})/H$:

$$\mathbb{C}^\times \times \bigl(\mathbb{C}^{n+1} \setminus \{0\}\bigr) / \ \{(t, \mathbf{w}) \sim (e^z t, e^{\alpha z}\mathbf{w})\} \cong \bigl(\mathbb{C}^{n+1} \setminus \{0\}\bigr) / \ \{\mathbf{w} \sim e^{2\pi i \alpha}\mathbf{w}\},$$

where $t \in \mathbb{C}^\times$, $\mathbf{w} \in \mathbb{C}^{n+1} \setminus \{0\}$. The Hopf manifold.

A holomorphic foliation on $\mathcal{Z}_\mathcal{K}$

Define the Lie subalgebra and the corresponding Lie group

$$\mathfrak{k} = \mathrm{Re}(\mathfrak{h}) \subset \mathbb{R}^m = \mathrm{Lie}(T^m), \qquad K = \exp(\mathfrak{k}) \subset T^m.$$

The restriction of the T^m-action on $U(\mathcal{K})/H$ to $K \subset T^m$ is almost free. Since $\mathfrak{h}_\mathbb{C} \cong \mathfrak{h} \oplus \mathfrak{k}$, we obtain a *holomorphic* foliation $\mathcal{F}$ on $\mathcal{Z}_\mathcal{K} = U(\mathcal{K})/H$ by the orbits of $K = H_\mathbb{C}/H$.

If the subspace $\mathfrak{k} \subset \mathbb{R}^m$ is rational (i. e., generated by integer vectors), then K is a subtorus of T^m and the complete simplicial fan $\Sigma := q(\Sigma_\mathcal{K})$ is rational. The rational fan Σ defines a toric variety

$$V_\Sigma = \mathcal{Z}_\mathcal{K}/K = U(\mathcal{K})/K_\mathbb{C}.$$

The holomorphic foliation of $\mathcal{Z}_\mathcal{K}$ by the orbits of K becomes a holomorphic Seifert fibration over the toric orbifold V_Σ with fibres compact complex tori $K_\mathbb{C}/H \cong T^{m-n}$.

The rational case:

$$
\begin{array}{ccccc}
\mathbb{C}^m \supset & & U(\mathcal{K}) & & \\
& & & \searrow{\scriptstyle H \cong \mathbb{C}^\ell} & \\
& {\scriptstyle K_{\mathbb{C}} \cong (\mathbb{C}^\times)^{m-n}} & \big\downarrow & & \mathcal{Z}_{\mathcal{K}} \quad \text{moment-angle manifold} \\
& & & \swarrow{\scriptstyle K_{\mathbb{C}}/H \cong T^{m-n}} & \\
\text{toric variety} & & V_\Sigma & &
\end{array}
$$

The non-rational case:

Have $U(\mathcal{K}) \xrightarrow{H} \mathcal{Z}_{\mathcal{K}}$,

and a holomorphic foliation $\mathcal{F}$ of $\mathcal{Z}_{\mathcal{K}}$ by the orbits of $K \subset T^m$.

The holomorphic foliated manifold $(\mathcal{Z}_{\mathcal{K}}, \mathcal{F})$ is a model for 'irrational' toric varieties in the sense of [Katzarkov, Lupercio, Meersseman, Verjovsky] (arXiv:1308.2774) and [Ratiu, Zung] (arXiv:1705.11110).

Basic cohomology

M a manifold with an action of a connected Lie group G, $\mathfrak{g} = \mathrm{Lie}\, G$.

$$\Omega(M)_{\mathrm{bas},\,G} = \{\omega \in \Omega(M)\colon \iota_\xi\omega = L_\xi\omega = 0 \text{ for any } \xi \in \mathfrak{g}\},$$

$H^*_{\mathrm{bas},\,G}(M) = H(\Omega(M)_{\mathrm{bas},\,G}, d)$ the basic cohomology of M.

$S(\mathfrak{g}^*)$ the symmetric algebra on $\mathfrak{g}^*$ with generators of degree 2.
The Cartan model is

$$\mathcal{C}_\mathfrak{g}(\Omega(M)) = ((S(\mathfrak{g}^*) \otimes \Omega(M))^\mathfrak{g}, d_\mathfrak{g}),$$

where $(S(\mathfrak{g}^*) \otimes \Omega(M))^\mathfrak{g}$ denotes the $\mathfrak{g}$-invariant subalgebra.
An element $\omega \in \mathcal{C}_\mathfrak{g}(\Omega(M))$ is a "$\mathfrak{g}$-equivariant polynomial map from $\mathfrak{g}$ to $\Omega(M)$". The differential $d_\mathfrak{g}$ is given by

$$d_\mathfrak{g}(\omega)(\xi) = d(\omega(\xi)) - \iota_\xi(\omega(\xi)).$$

Theorem

$$H^*_{\mathrm{bas},\,G}(M) \cong H\bigl(\mathcal{C}_{\mathfrak{g}}(\Omega(M)), d_{\mathfrak{g}}\bigr).$$

If in addition G is a compact, then

$$H^*_{\mathrm{bas},\,G}(M) \cong H^*_G(M) = H^*(EG \times_G M) \quad \text{the equivariant cohomology.}$$

Now consider $\mathcal{Z}_{\mathcal{K}}$ with the action of K (the holomorphic foliation $\mathcal{F}$).

Theorem (Ishida–Krutowski–P.)

There is an isomorphism of algebras:

$$H^*_{\mathrm{bas},\,K}(\mathcal{Z}_{\mathcal{K}}) \cong \mathbb{C}[v_1, ..., v_m]/(I_{\mathcal{K}} + J_{\Sigma}),$$

where $I_{\mathcal{K}}$ is the Stanley–Reisner ideal of $\mathcal{K}$, generated by the monomials

$$v_{i_1} \cdots v_{i_k} \quad \text{with } \{i_1, \ldots, i_k\} \notin \mathcal{K},$$

and J_{Σ} is the ideal generated by the linear forms

$$\sum_{i=1}^{m} \langle \mathbf{a}_i, \mathbf{u} \rangle v_i \quad \text{with } \mathbf{u} \in (\mathbb{R}^m/\mathfrak{k})^*.$$

This settles a conjecture by [Battaglia and Zaffran] (arXiv:1108.1637).

If K is a compact torus (the fan Σ is rational), then we get

$$H^*_{\mathrm{bas},\,K}(\mathcal{Z}_\mathcal{K}) = H^*(\mathcal{Z}_\mathcal{K}/K) = H^*(V_\Sigma)$$

and the result above turns into the well-known description of the cohomology of toric manifolds, due to [Danilov and Jurkiewicz].

Idea of proof of the theorem.

Let $\mathfrak{t} = \mathrm{Lie}(T^m) \cong \mathbb{R}^m$ and consider the Cartan model

$$\mathcal{C}_\mathfrak{t}(\Omega(\mathcal{Z}_\mathcal{K})) = \big((S(\mathfrak{t}^*) \otimes \Omega(\mathcal{Z}_\mathcal{K}))^{T^m}, d_\mathfrak{t}\big).$$

Then

$$H(\mathcal{C}_\mathfrak{t}(\Omega(\mathcal{Z}_\mathcal{K}))) = H^*_{T^m}(\mathcal{Z}_\mathcal{K}) = \mathbb{C}[v_1, \ldots, v_m]/I_\mathcal{K}.$$

Key lemma: the dga $\mathcal{C}_\mathfrak{t}(\Omega(\mathcal{Z}_\mathcal{K}))$ is formal (quasi-isomorphic to its cohomology). □

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