On a variant of the Rudin–Keisler preorder defined via finitary maps

N. L. Polyakov, Higher School of Economics
D. I. Saveliev, Russian Academy of Sciences

The classical notion of the Rudin–Keisler preorder on ultrafilters over an infinite set X is defined by letting $u \leq_{RK} v$ iff there exists a map $f : X \to X$ such that $\tilde{f}(u) = v$ for all $u, v \in \beta X$, where βX is the set of ultrafilters over X endowed with the standard topology, and $\tilde{f} : \beta X \to \beta X$ is the continuous extension of f (see e.g. [1, 2]). By using right continuous extensions of maps of any finite arities (introduced in [3, 4]), we expand this notion as follows. Let uR_nv iff there exists an n-ary map $f : X^n \to X$ such that $\tilde{f}(u, \ldots, u) = v$ for all $u, v \in \beta X$ where $\tilde{f} : (\beta X)^n \to \beta X$ is the right continuous extension of f. Let also $\leq$ be the union of all the R_n, $n \in \omega$. Clearly, R_1 coincides with $\leq_{RK}$, and $R_m \subseteq R_n$ whenever $m \leq n$.

We show that $R_m \circ R_n = R_{m \cdot n}$, whence it follows that $\leq$ is a preorder. We prove that this preorder properly extend the Rudin–Keisler preorder, i.e., there are $u, v \in \beta X$ such that $u \leq v$ but not $u \leq_{RK} v$. As well-know, under the Continuum Hypothesis (CH) there exist non-principal ultrafilters over ω that are $\leq_{RK}$-minimal (see [1]). We show that an analogous fact is true for the stronger preorder $\leq$. To prove this, we state the following generalization of the Ramsey theorem, which may be of an independent interest: for any $n \in \omega$ and n-ary map $f : \omega^n \to \omega$ there exist $m \leq n$, $K \in [n]^m$, and an infinite $A \subseteq \omega$ such that for increasing n-tuples $x_0 < \ldots < x_{n-1}$ and $y_0 < \ldots < y_{n-1}$ in A we have $f(x_0, \ldots, x_{n-1}) = f(y_0, \ldots, y_{n-1})$ iff $x_k = y_k$ for all $k \in K$. (The case $m = 0$ gives a constant map.)

Finally, we expand the hierarchy from the relations R_n to R_α for arbitrary ordinals α, and strengthen some of obtained results. In particular, we show that R_α are increase by inclusion as α grows, and included in the Comfort preorder. On $\beta\omega$, all R_α with $\alpha < \omega_1$ are distinct, and $R_{<\alpha} = \bigcup_{\xi<\alpha} R_\xi$ is a preorder whenever α is a multiplicatively indecomposable ordinal.

References

[1] W. W. Comfort, S. Negrepontis, The theory of ultrafilters, Springer, Berlin, 1974.

[2] N. Hindman, D. Strauss, Algebra in the Stone–Cech compactification, 2nd ed., revised and expanded, W. de Gruyter, Berlin–N.Y., 2012.

[3] D. I. Saveliev, Ultrafilter extensions of models, Lecture Notes in AICS 6521 (2011), 162–177.

[4] D. I. Saveliev, On ultrafilter extensions of models, in: S.-D. Friedman et al. (eds.), The Infinity Project Proc., CRM Documents 11, Barcelona, 2012, 599–616.

The second author was supported by the RFBR grant 17-01-00705.